Summary

Thm Let E be an algebraic extension of F (a field).

Then $[E:F] < \infty \iff \exists\, \alpha_1, \dots, \alpha_k$ such that $E = F(\alpha_1, \dots, \alpha_k)$.

Pf. Take $\alpha_1 \in E \setminus \{F\}$. $\Rightarrow F(\alpha_1)$ has finite index over F.

$\therefore [E : F(\alpha_1)] < [E:F]$.

Keep adjoining $\alpha_2, \alpha_3, \dots$ until $[E : F(\alpha_1, \dots, \alpha_k)] = 1$

$\Rightarrow E = F(\alpha_1, \dots, \alpha_k)$. $\blacksquare$

Thm: Suppose a field F has an extension field E.

We define $\overline{F_E} = \{\alpha \in E : \alpha \text{ is alg. over } F\}$.

Then $\overline{F_E}$ is a subfield of E.

($\overline{F_E}$ is called the algebraic closure of F in E.)

Pf. Suppose $\alpha, \beta \in \overline{F_E}$. Then $F(\alpha, \beta)$ is a finite extension of $F \Rightarrow$ it's an algebraic extension. (So every $\gamma \in F(\alpha,\beta)$ is alg. over $F \Rightarrow \gamma \in \overline{F_E}$.) $\Rightarrow \overline{F_E}$ contains $\alpha+\beta, \alpha-\beta, \alpha\beta, \frac{\alpha}{\beta}$ if $\beta \neq 0$.

$\Rightarrow \overline{F_E}$ is a subfield of E. $\blacksquare$

Cor: The set of algebraic numbers is a field extension of $\mathbb{Q}$, a subfield of $\mathbb{C}$.

$\overline{\mathbb{Q}_{\mathbb{C}}}$.

Defn: A field F is called *algebraically closed* if every $f(x) \in F[x]$ has a root in F.

($\Rightarrow f(x)$ splits in F.)

Cor. If F is algebraically closed, there does not exist a proper algebraic extension E of F.

(e.g. no finite extensions)

Pf. Suppose $\alpha \in E$. $\Rightarrow \text{irr}(\alpha, F) = x - \alpha$.

$\Rightarrow [F(\alpha) : F] = 1 \Rightarrow \alpha \in F$. $\times$.

unless $E = F$.

Thm Every field has an algebraic closure

(ie if F is a field, $\exists$ extension E of F s.t. E is alg. closed.)

Pf. — utilizes Zorn's Lemma (equivalent to the axiom of choice).

Use Kronecker's Thm $\leftarrow$ use infinite induction.

Thm $\mathbb{C}$ is algebraically closed.

Pf. [There exists an algebraic that doesn't use complex analysis or topology, but it is very long.]

Suppose not, ie. $\exists f(x) \in \mathbb{C}[x]$ with no zero in $\mathbb{C}$, $f(x) \neq$ constant. Note f is holomorphic & has no zeros

$\Rightarrow \frac{1}{f(z)}$ is also holomorphic on $\mathbb{C}$.

$a_n \neq 0$

let $f(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_1 z + a_0$

$$\lim_{|z|\to\infty} \frac{1}{|f(z)|} = \lim_{|z|\to\infty} \frac{1}{|a_n||z|^n \underbrace{\left|1 + \frac{a_{n-1}}{|a_n| z} + \frac{a_{n-2}}{|a_n| z^2} + \dots + \frac{a_0}{|a_n| z^n}\right|}_{\downarrow \; 1}}$$

$$= 0.$$

$\left|\frac{1}{f(z)}\right| < 1$ $\quad \Rightarrow$ for $\varepsilon = 1$, $\exists R > 0$ s.t. $|z| > R$

$$\Rightarrow \frac{1}{|f(z)|} < \varepsilon = 1$$

$$\therefore \left|\frac{1}{f(z)}\right| \le \underbrace{\max_{z \in B_R(0)} \left|\frac{1}{f(z)}\right|}_{\text{finite.}} + 1$$

or. $\left|\frac{1}{f(z)}\right| \le \max\{M, 1\}$.

$\therefore$ $\frac{1}{f(z)}$ is a holom. fcn that is bounded in $|\;|$ over all of $\mathbb{C}$.

Liouville's Thm $\Rightarrow \frac{1}{f(z)} = C \Rightarrow f(z) = \frac{1}{C} =$ constant.

$\times$.

$\therefore$ No such $f(z)$ exists.

$\Rightarrow \mathbb{C}$ is alg. closed. $\boxtimes$

Liouville Thm.: An entire holom. fcn that is bdd on $\mathbb{C}$ is constant.

Proof Liouville Thm.

Let f be an entire bdd fcn.

$$\Rightarrow f'(z) = \frac{1}{2\pi i} \int_{C_R(z)} \frac{f(a)}{(a-z)^2}\, da \quad \text{by CIF for derivatives.}$$

$$\Rightarrow |f'(z)| \le \frac{1}{2\pi} \int_{C_R(z)} \left|\frac{f(a)}{(a-z)^2}\right| |da|$$

say $|f(a)| \leq K$

$$\leq \frac{1}{2\pi} \int_{C_R(z)} \frac{K}{R^2} |da|$$

$$\int \frac{f(\xi)\, d\xi}{\xi - z_0}$$

$$= 2\pi R \left(\frac{1}{2\pi} \frac{K}{R^2} \right)$$

$$\Rightarrow |f'(z)| \leq \frac{K}{R} \longrightarrow 0 \text{ as } R \to \infty$$

independent of R.

$$\Rightarrow f'(z) = 0 \Rightarrow f(z) \text{ is constant.}$$

$\boxed{\checkmark}$